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# Recent Advances in Nuclear Reaction Theories for Weakly Bound Nuclei: Reexamining the Problem of Inclusive Breakup

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**Abstract** The problem of the calculation of inclusive breakup cross sections in nuclear reactions is reexamined. For that purpose, the theory proposed by Ichimura et al. (Phys Rev C 32:431, 1985) is revisited, both in its prior and post representations. We briefly outline the connection of this theory with that proposed by Udagawa and Tamura (Phys Rev C 24:1348, 1981) and apply both theories to the inclusive breakup of $^{6}$Li on $^{209}$Bi at near-barrier energies, comparing also with available data. The relative importance of elastic versus non-elastic breakup, as a function of the incident energy and of the projectile separation energy, is also investigated.

## 1 Introduction

Weakly-bound nuclei are known to break easily in collisions with other nuclei. Breakup reactions induced by weakly-bound projectiles have been used to extract nuclear structure information (binding energies, spectroscopic factors, electric response to the continuum, etc) and have also permitted to improve our understanding of the dynamics of reactions among composite systems. Furthermore, it is long known that in these reactions breakup channels may have a strong influence on other channels, such as elastic scattering [2,15,20,47], transfer [8,37] and fusion [9,16,22].

Considering for simplicity the case of two-body breakup, these reactions can be schematically represented as $a + A \to b + x + A$, where $a = b + x$ represents the two-body projectile. Even in this case, the theoretical description of the process is not straightforward due to the presence of three particles in the final state.

Experimentally, two distinct situations can be distinguished. The first one is that in which the state of the three outgoing fragments is fully determined (i.e., measured). In this case, the reaction is said to be *exclusive*. The second situation is that in which the state of one or more particles is not fully specified in the exit channel. This occurs, for example, when one or more particles are not detected in the experiment. In this case, the reaction is said to be *inclusive* with respect to unobserved fragment(s).

From the theoretical point of view, exclusive breakup reactions are well understood and a variety of theories are nowadays available to compute the corresponding cross sections. These theories usually treat the reaction assuming an effective three-body scattering problem, with some effective two-body interactions. Although the rigorous formal solution of this problem is given by the Faddeev formalism [25,27], the difficulty of solving these equations has led to the development of simpler approaches, such as the distorted-wave Born approximation (DWBA) [4], the continuum-discretized coupled-channels (CDCC) method [2] and a variety of

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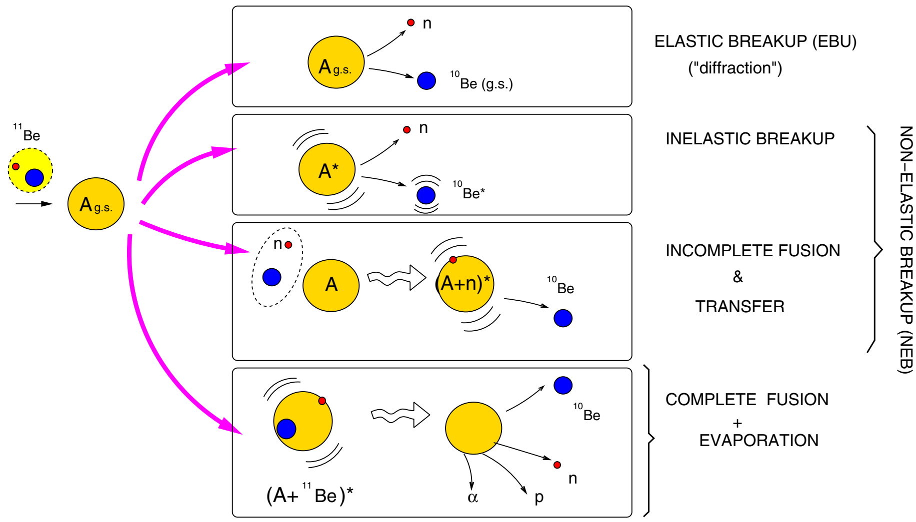


**Fig. 1** (*Color online*) Two-body breakup modes for the $^{11}$Be+A reaction

semiclassical approaches [10,24,39,57]. Recently, it has also become possible to solve the Alt-Grassenberg-Grandas (AGS) formulation of the Faddeev equations for specific cases [18,19].

A difficulty inherent to the theoretical description of inclusive reactions is that they involve a sum over all possible final states of the unobserved particle(s). For example, using the notation introduced above, and assuming that only $b$ is observed, the reaction can be represented as $a + A \rightarrow b + X$, where $X$ is any possible configuration of the $x + A$ system. The main contributing processes will be the following:

(i) The *elastic breakup* process (EBU), in which the three outgoing particles are emitted in their ground state, i.e., $a + A \rightarrow b + x + A_{\rm gs}$.
(ii) Inelastic breakup (INBU), in which the breakup is accompanied by the excitation of some of the fragments. For example, if the target is excited, $a + A \rightarrow b + x + A^*$, whereas if the core particle is excited, $a + A \rightarrow b^* + x + A_{\rm gs}$.
(iii) Particle transfer, leading to bound states of the $A + x$ system, i.e. $a + A \rightarrow b + B$ ($B \equiv A + x$).
(iv) Incomplete fusion (ICF), in which the fragment $x$ is absorbed by the target, forming a compound nucleus $C$, which will eventually decay by particle or gamma-ray emission: $a + A \rightarrow b + C$.
(v) Complete fusion (CF) followed by evaporation. If $b$ is among the evaporation products, it will contribute also to the inclusive $b$ yield. We include also in this category the preequilibrium (PE) processes.

In Fig. 1, these processes are schematically depicted for a $^{11}$Be+A reaction (assuming the two-body dissociation $^{11}\text{Be} \rightarrow ^{10}\text{Be} + n$).

The EBU cross sections [process (i)], can be accurately obtained with the three-body models cited above, either quantum-mechanical (DWBA, CDCC, AGS/Faddeev) or semiclassical.

The calculation of INBU, process (ii), has been less explored in the literature. In the case of target excitation, this was done by the Kyushu group in the early days of the CDCC method [63] for the case of deuteron scattering, with the aim of comparing the relative importance and mutual influence of target-excitation and deuteron breakup in elastic and inelastic scattering of deuterons. In these calculations, the target excitation was treated within the vibrational model. Although in the cases investigated by these authors the target excitation effect was found to be relatively small, it would be of interest to revisit and implement the formalism with the aim of applying it to the interpretation of new inelastic-scattering measurements induced by weakly-bound projectiles.

The inclusion of excitations of the projectile constituents ($b$ or $x$ in our case) has not been implemented in the CDCC method until very recently. This has been done using a no-recoil DWBA model (XDWBA) [14,44], and also using an extended version of the CDCC method (XCDCC) [17,54]. It is worth noting that these core excitations effects will influence both the projectile structure and the reaction dynamics. In the inert-core picture assumed by standard three-body models, the projectile states are represented by pure single-particle or cluster states. However, if the core is allowed to excite, the projectile states will contain in general admixtures of these core excited components. The dynamical effect arises due to the excitations of the $b$ core due to its interaction with the target. These collective excitations will compete and even interfere [45] with the valence excitation mechanism.

Calculations using the XCDCC method were first performed by Summers et al. for $^{11}$Be and $^{17}$C on $^{9}$Be [54] and $^{11}$Be+$p$ [53]. Later on, De Diego et al. [17] performed calculations for $^{11}$Be on protons, $^{64}$Zn and $^{208}$Pb. These calculations have shown that dynamic core excitations tend to increase significantly the breakup cross sections for the lighter targets whereas, for the heavier targets, the dynamic core excitation mechanism is small, although the effect on the projectile strucutre can be very important.

Process (iii), i.e., transfer of $x$ to bound states of $A$, has been traditionally treated within the DWBA method [51]. For weakly-bound projectiles, the coupling to the breakup channels becomes important, and this effect is known to affect the transfer cross sections. This effect can be incorporated using the adiabatic distorted wave model of Johnson and Soper (ADWA) [37] and more elaborate versions of it (e.g. [36]). A recent review of these theories can be found in Ref. [8].

The process (iv), ICF, is very challenging from the theoretical point of view to the extent that, at present, no fully-quantum mechanical theory exists to calculate ICF cross sections. For this reason, alternative methods, based on semiclassical ideas, have been proposed in the literature [21,23,43]. Moreover, from the experimental point of view, the identification of this process is not without its difficulties since, many times, the products coincide with those produced in the transfer reactions.

Processes (ii)–(iv) will be henceforth referred to as non-elastic breakup (NEB). The theoretical evaluation of NEB cross sections is the main topic of this contribution.

Process (v) is qualitatively different from the previous ones, because it takes place via the formation of a compound nucleus, rather than via a direct process. The calculation of detailed cross sections, as a function of the angle/energy of the outgoing particles, requires the use of statistical models, first proposed by Bohr [6], and whose modern formulation can be found in many textbooks [56].

According to the previous discussion the total singles cross section for the production of $b$ fragments can be written as

$$\sigma_b = \sigma_{\mathrm{CF}}^{(b)} + \sigma_{\mathrm{EBU}} + \sigma_{\mathrm{NEB}}. \tag{1}$$

where $\sigma_{\mathrm{CF}}^{(b)}$ is the part of the CF cross section evaporating $b$ particles.

Due to the large number of accessible states, a detailed calculation of the NEB part, in which all these processes are included explicitly, is in general not possible. This led in the 1980s to the development of alternative methods, in which this sum was reduced, after some approximations, to a closed form. For example, in the pioneering works by Baur and co-workers [3,7,52], the sum was done making use of unitarity and a surface approximation of the form factors for the excited states of the residual nucleus. These two approximations were avoided in later works by Udagawa and Tamura [58,59], who used a prior-form DWBA formalism, and by Austern and Vincent [1], who used the post-form DWBA. Starting from this latter model, Kasano and Ichimura [38] found a formal separation between the EBU and NEB contributions. These results were carefully reviewed by Ichimura, Austern and Vincent [33] and the model was subsequently referred to as the IAV formalism.

The problem of the evaluation of NEB has received a renewal interest in recent years [11,41,46]. All these recent works have made use of the IAV model, either in its original post-form representation [11,41] or in its equivalent prior-form [46]. In the next Section, we briefly recall this model and present some applications to $^{6}$Li inclusive breakup, comparing with available data.

## 2 The Ichimura, Austern, Vincent (IAV) Model for Non-elastic Breakup

The theory proposed by IAV is based on the participant-spectator model [30]. Using the notation introduced in the previous section, the particle $b$ is treated as an *spectator*, meaning that its interaction with the target nucleus is described with an optical potential $U_{bA}$. Thus, possible excitations of $A$ due to its interaction with $b$ are encoded in the imaginary part of this potential. On the other hand, the interaction of the *participant* particle $x$ with the target is described with the microscopic potential $V_{xA}$, which retains its dependence on the target degrees of freedom. Consequently, the process is studied with the effective Hamiltonian

$$H = K + V_{bx}(\mathbf{r}_{bx}) + U_{bA}(\mathbf{r}_{bA}) + H_A(\xi) + V_{xA}(\xi, \mathbf{r}_x), \tag{2}$$

where $K$ is the total kinetic energy operator, $V_{bx}(\mathbf{r}_{bx})$ is the interaction binding the clusters $b$ and $x$ in the initial composite nucleus $a$ and $H_A(\xi)$ is the Hamiltonian of the target nucleus (with $\xi$ denoting its internal coordinates). The relevant coordinates are depicted in Fig. 2.

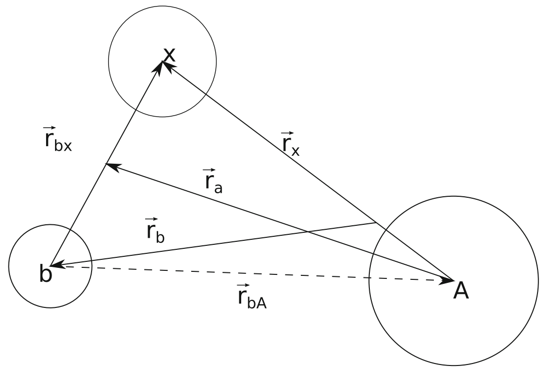


**Fig. 2** Relevant coordinates used in the IAV model described in the text

Using the post-form representation of the transition amplitude, the inclusive breakup differential cross section, as a function of the detected angle and energy of the fragment $b$, is given by (see, e.g. Ref. [2])

$$\frac{d^2\sigma}{dE_b d\Omega_b} = \frac{2\pi}{\hbar v_a}\rho(E_b)\sum_c |\langle \chi_b^{(-)}\Psi_{xA}^{c,(-)}|V_{\rm post}|\Psi^{(+)}(\xi, \mathbf{r}_{bx}, \mathbf{r}_a)\rangle|^2\delta(E - E_b - E^c), \tag{3}$$

where $V_{\rm post} \equiv V_{bx} + U_{bA} - U_{bB}$ is the post-form transition operator,[1] $v_a$ the projectile-target relative velocity in the incident channel, $\rho_b(E_b) = k_b\mu_b/[(2\pi)^3\hbar^2]$ is the so-called density of states of $b$ (with $\mu_b$ the reduced mass of $b + B$ and $k_b$ their relative wave number), $\Psi^{(+)}$ is the exact (and hence unknown) wave function of the problem, $\chi_b^{(-)}(\mathbf{k}_b, \mathbf{r}_b)$ is the distorted wave describing the $b - B$ relative motion, and generated with the optical potential $U_{bB}$, and $\Psi_{xA}^{c,(-)}$ are the wave functions describing the states of the $x + A$ system, with $c = 0$ denoting the $x$ and $A$ ground states. Thus, the $c = 0$ and $c \neq 0$ terms correspond, respectively, to the EBU and NEB contributions defined in the Introduction.

In actual calculations the exact wave function $\Psi^{(+)}$ must be approximated somehow. For example, in DWBA, one assumes the factorized form[2]

$$\Psi^{(+)}(\xi, \mathbf{r}_x, \mathbf{r}_b) \approx \phi_A^0(\xi)\phi_a(\mathbf{r}_{bx})\chi_a^{(+)}(\mathbf{k}_a, \mathbf{r}_a), \tag{4}$$

where $\phi_a(\mathbf{r}_{bx})$ and $\phi_A^0(\xi)$ are the projectile and target ground-state wave functions, and $\chi_a^{(+)}$ is a distorted wave describing the $a + A$ motion in the incident channel.

The NEB cross section can be interpreted as the absorption in the $x + A_{\rm gs}$ channel, that is, the flux leaving this channel. Following our previous work [41], this absorption cross section can be evaluated making use of the *generalized optical theorem* [13] and taking into account the density of final states for the $b$ particle. This gives the following expression for the double differential cross sections of detected $b$ particles

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\rm NEB}^{\rm IAV} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\rm post}(\mathbf{k}_b)|W_x|\psi_x^{\rm post}(\mathbf{k}_b)\rangle, \tag{5}$$

where $\psi_x^{\rm post}$ is the $x$-channel wave function describing the evolution of the $x$ particle after dissociating from the $a$ nucleus, for a given final state of the $b$ particle (characterized by its momentum $\mathbf{k}_b$) and projected onto the target ground state. Austern and Vincent [1] showed that this channel wave function is the solution of the inhomogeneous equation

$$(E_x^+ - K_x - U_x)\psi_x^{\rm post}(\mathbf{k}_b, \mathbf{r}_x) = (\chi_b^{(-)}(\mathbf{k}_b)|V_{\rm post}|\phi_a\chi_a^{(+)}\rangle, \tag{6}$$

where $E_x^+ = E_x + i\epsilon$ ($\epsilon \to 0$), with $E_x = E - E_b$. The notation $(||\rangle$ denotes integration over $\mathbf{r}_b$ only.

[1] In their original papers [33], IAV usually make the approximation $V_{\rm post} \approx V_{bx}$, thus neglecting the so-called remnant term, $U_{bA} - U_{bB}$. In Ref. [41] we showed that this is a good approximation for deuterons on heavy targets, but not for $^6$Li reactions. In the calculations presented in this work we retain the full transition operator.

[2] Note that this wave function will depend also on the incident momentum $\mathbf{k}_a$, but this dependence will be omitted for simplicity of the notation.

The IAV model has been recently revisited by several groups [11,41,46]. In Refs. [11,46], the theory was applied to deuteron induced reactions of the form $A(d, pX)$ and in Ref. [41] the model was extended to $^6$Li induced reactions of the form $A(^6\text{Li},\alpha X)$. In general, the agreement with the data has been found to be very promising and several extensions and improvements are underway (see Sec. 5).

## 3 Post-prior Equivalence of the NEB Cross Sections

It is well known that the direct evaluation of the post-form breakup transition amplitude is not feasible because it involves slow-converging integrals. In the present context, this problem arises also in the solution of Eq. (6), owing to the oscillatory behavior of the distorted waves $\chi_b^{(-)}$ in the source term. To overcome this problem, several regularization procedures have been proposed in the literature, such as the complex-plane integration of Vincent and Fortune [61], the introduction of a convergence damping factor [29,60] or the replacement of the oscillatory distorted waves $\chi_b^{(-)}$ by some average wave packets [55]. Alternatively, one may reformulate the problem using the prior-form representation of the transition amplitude. For transfer reactions between bound states, it is well known that the post and prior formulas are equivalent. Moreover, in DWBA, the post and prior expressions are formally identical, differing only in the use of the post or prior transition potential, respectively. IAV [33] derived the prior-form expression of their model, and demonstrated that the post-prior equivalence does also hold for the NEB cross sections. However, unlike the case of transfer reactions, the post and prior expressions are not formally identical.

The connection between these two representations can be derived using the following relation obtained in Ref. [42]

$$\psi_x^{\text{post}} = \psi_x^{\text{prior}} + \psi_x^{\text{NO}}, \tag{7}$$

where $\psi_x^{\text{prior}}$ is the prior-form $x$-channel wave function, which is a solution of

$$(E_x^+ - K_x - U_x)\psi_x^{\text{prior}}(\mathbf{k}_b, \mathbf{r}_x) = (\chi_b^{(-)}(\mathbf{k}_b)|V_{\text{prior}}|\chi_a^{(+)}\phi_a\rangle, \tag{8}$$

with $V_{\text{prior}} \equiv U_{xA} + U_{bA} - U_{aA}$, and

$$\psi_x^{\text{NO}}(\mathbf{k}_b, \mathbf{r}_x) = (\chi_b^{(-)}(\mathbf{k}_b)|\chi_a^{(+)}\phi_a\rangle, \tag{9}$$

is the so-called non-orthogonality NO overlap.

Replacing (7) into Eq. (5) one gets

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{IAV}} = \left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{UT}} + \left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{NO}} + \left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{IN}}, \tag{10}$$

where

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{UT}} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\text{prior}}(\mathbf{k}_b)|W_x|\psi_x^{\text{prior}}(\mathbf{k}_b)\rangle, \tag{11}$$

is the Udagawa and Tamura (UT) formula (see discussion below) [58],

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{NO}} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\text{NO}}(\mathbf{k}_b)|W_x|\psi_x^{\text{NO}}(\mathbf{k}_b)\rangle \tag{12}$$

is the non-orthogonality (NO) *cross section* and

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\text{NEB}}^{\text{IN}} = -\frac{4}{\hbar v_a}\rho_b(E_b)\text{Re}\langle\psi_x^{\text{prior}}(\mathbf{k}_b)|W_x|\psi_x^{\text{NO}}(\mathbf{k}_b)\rangle \tag{13}$$

is the interference (IN) term.

Equation (10) represents the post-prior equivalence of the NEB cross sections in the IAV model, with the RHS corresponding to the prior-form expression of this model. Interestingly, the first term corresponds to the NEB formula proposed by Udagawa and Tamura [58]. It is analogous to the IAV post-form formula (5), but with the $x$-channel wave function replaced by $\psi_x^{\text{prior}}(\mathbf{k}_b, \mathbf{r}_x)$. However, the prior-form formula of

IAV contains two additional terms. These terms ensure the post-prior equivalence of the NEB cross sections. UT considered that the latter are unphysical and hence that the post-prior equivalence does not hold for the NEB. This discrepancy led to a long-standing controversy between these two groups, which lasted for more than a decade. The problem was later addressed in subsequent works by Ichimura et al. [32,34,35] and also by Hussein and co-workers [31]. These works clearly demonstrated that the UT formula provides only the so-called *elastic breakup fusion* component, which corresponds to breakup without simultaneous excitation of the target $A$ by the interaction $V_{xA}$, and that the prior-post equivalence does indeed hold for inclusive processes as well. The problem has been also recently reexamined [40,46] and the calculations performed in these works seem to corroborate the validity of the IAV model over the UT model. We note here that this problem does not arise for the EBU part, for which the post and prior formulas are well known to give identical results [33].

## 4 Application of the IAV Model to $^6$Li-Induced Reactions

In this section, we apply the IAV formalism to the $^6$Li+$^{209}$Bi reaction at several incident energies. This reaction was also studied in our previous work [41], where we showed that this model reproduces rather well the experimental angular distributions of $\alpha$ particles for a wide range of incident energies above and below the Coulomb barrier. For completeness, we present here the results at two incident energies, $E = 30$ MeV and $E = 38$ MeV, along with the corresponding experimental data. These results are displayed in Fig. 3. The left panels contain the elastic angular distributions (relative to Rutherford) at these two incident energies. The data from Ref. [48] are compared with CDCC calculations (assuming a $\alpha + d$ structure model for $^6$Li) and OM calculations. For the later, we used the global potential of Cook [12]. The details of the CDCC calculations (continuum discretization, optical potentials, etc) can be found in Ref. [41]. It is seen that both calculations reproduce very well the data at both energies. Yet, it is important to note that, in the CDCC calculations, the imaginary part of the deuteron–target potential needed to be reduced for a correct reproduction of the data. This reduction of the imaginary part has been found necessary in other CDCC analyses of $^6$Li elastic scattering data (see, e.g. Refs. [5,28,50]) and has been recently attributed to the limitation of the two-body description of the $^6$Li nucleus [62].

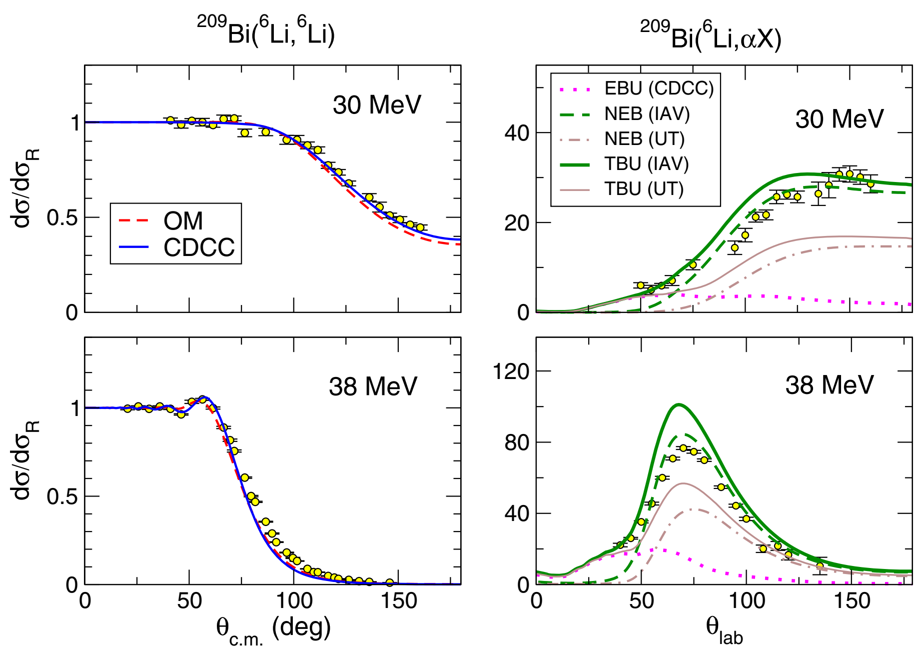


**Fig. 3** (*Color online*) *Left*: Elastic angular distribution, relative to Rutherford, for $^6$Li+ $^{209}$Bi at 30 MeV (*top*) and 38 MeV (*bottom*). The *solid* and *dashed lines* are, respectively, the CDCC and OM calculations, with the latter using the optical potential from Ref. [12]. *Right*: Angular distribution of $\alpha$ particles (in the laboratory frame) produced in the same reaction. The *dotted line* is the EBU contribution computed with CDCC, the *dashed line* is the NEB result obtained with the IAV model, and the *thick solid line* represents their incoherent sum (TBU = EBU + NEB). The *dot-dashed* and the *thin solid lines* are the NEB and TBU calculated with the UT model. The experimental data are from Refs. [48,49]

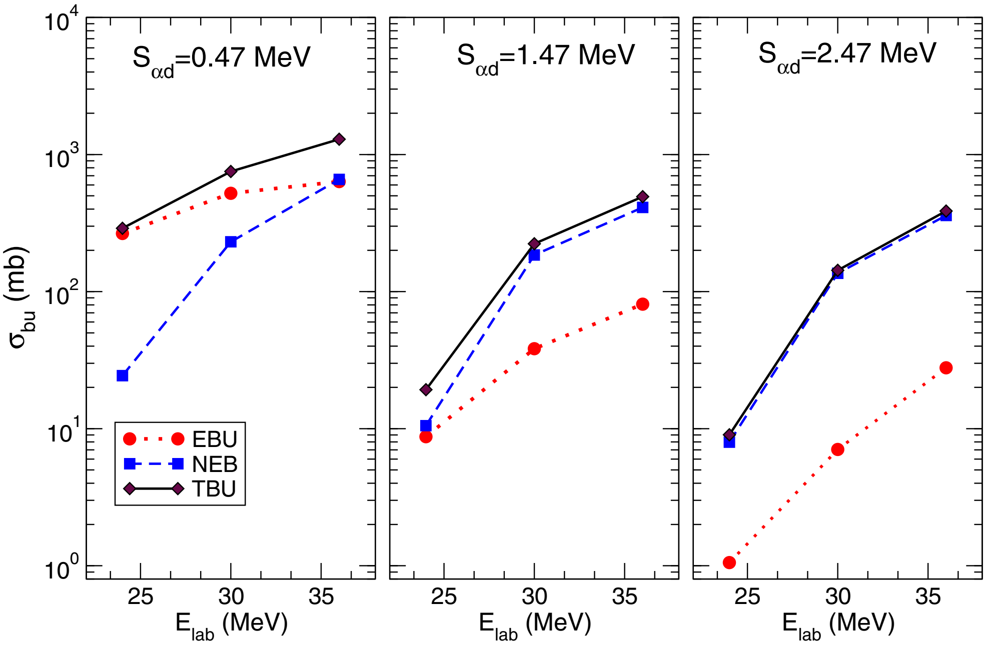


**Fig. 4** (*Color online*) Elastic (EBU), non-elastic (NEB) and total breakup (TBU = EBU + NEB) cross sections for the $^{209}$Bi($^{6}$Li,$\alpha X$) reaction as a function of the $^{6}$Li incident energy, and for three different values of the separation energy of $^{6}$Li

The right panel of Fig. 3 shows the experimental [49] and calculated angular distribution of $\alpha$ particles. For the calculations, we have considered both the EBU and NEB contributions. Possible contributions coming from CF followed by $\alpha$ evaporation are neglected. Thus, the total breakup (TBU) is assumed to be well approximated by the sum TBU = EBU + NEB. The EBU was obtained from the aforementioned CDCC calculations. For the NEB, we use the post-form DWBA version of the IAV model [c.f. Eq. (5)]. Notice that, in order to get converged results of the post-form formula, the distorted waves $\chi_b^{(-)}$ were averaged over small momentum *bins* [41]. It is seen that the sum EBU+NEB reproduces reasonably well the data, except for some overestimation of the magnitude. It is also observed that the inclusive $\alpha$ yield is largely dominated in this case by the NEB component. We have also included in this figure the prediction of the UT model, given by the first term in Eq. (10) (dot-dashed line). The TBU cross section obtained with this model (thin solid line) clearly underestimates the data. Consistently with recent works [11,41,46], this result supports the IAV model over the UT model.

Despite the clear dominance of NEB in the discussed cases, it is expected that the relative importance of EBU versus NEB will depend on several factors, such as the target mass/charge, the separation energy of the projectile and the incident energy. For example, in the scattering of neutron-halo nuclei on heavy targets, long-range Coulomb couplings favor the distant breakup of the projectile thus enhancing the EBU component over the NEB one. The effect has been found to be particularly remarkable at energies around and below the Coulomb barrier [20,26]. This effect is not observed in the $^{6}$Li case because dipole Coulomb couplings are suppressed due to the vanishing effective charge of this nucleus. One may nevertheless expect that, as the incident energy decreases, the EBU component will become progressively more important as compared to the NEB part, because the breakup will occur at larger distances, thus suppressing the absorption of the $d$+target system. We have studied this dependence by performing calculations at three incident energies, one below, one around and one above the Coulomb barrier ($V_b \sim 30.1$ MeV). Simultaneously, we have also studied the dependence on the binding energy by varying artificially the separation energy of the $^{6}$Li nucleus ($S_{\alpha d}$). The results are presented in Fig. 4 for the $^{209}$Bi($^{6}$Li,$\alpha X$) reaction. The left, middle and right panels correspond to the binding energies $S_{\alpha d} = 0.47, 1.47$ (the physical one) and 2.47 MeV, respectively. In each panel, we display the EBU, NEB and TBU cross sections as a function of the incident energy.

It is seen that the EBU depends strongly on the separation energy, decreasing by ~1-2 orders of magnitude when the latter is artificially increased from 0.47 to 2.47 MeV. By contrast, the NEB breakup shows a moderate reduction with this increase of binding energy. As a consequence, the relative importance of EBU versus NEB varies drastically with the separation energy. For $S_{\alpha d} = 1.47$ and 2.47 MeV, the TBU is largely dominated by the NEB component, whereas for $S_{\alpha d} = 0.47$ MeV (typical of halo nuclei), the EBU component dominates.

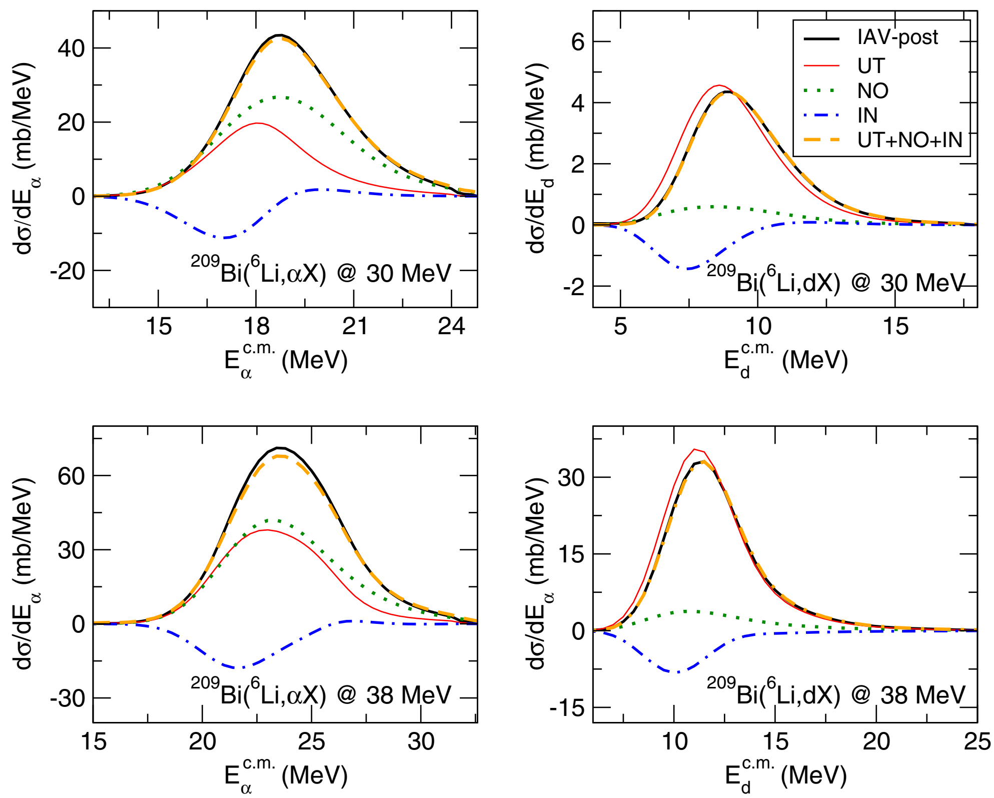


**Fig. 5** (*Color online*) Energy distribution of $\alpha$ particles (*left*) and deuterons (*right*) produced in the $^{6}$Li+$^{209}$Bi reaction at two different incident energies. The *thick solid line* is the result of the post-form IAV formula. The *thin solid*, *dotted* and *dot-dashed lines* are the UT, NO and IN contributions of the prior-form expression. The *dashed line* is the sum of these three contributions, i.e., the prior-form IAV result

This different behavior of the EBU and NEB components can be understood as follows. The EBU is a peripheral process and thereby highly sensitive to the tail of the $\alpha - d$ wave function. Since the magnitude of the wave function at large distances is mostly determined by the separation energy of the two clusters, it is conceivable that the EBU is reduced as the binding energy is increased. On the contrary, Eq. (5) indicates that the NEB component depends on the internal region, and will be therefore sensitive to the overall size of the projectile and target, being therefore less sensitive to the change in the tail of the $\alpha - d$ relative wave function.

Regarding the dependence on the incident energy we see in Fig. 4 that, for the physical separation energy (middle panel), the NEB largely dominates at energies around and above the barrier, and the EBU only becomes competitive at energies well below the barrier, for which the breakup is expected to occur at large projectile-target separations, and the absorptive effect of the $d$+target interaction will be less effective. For the more weakly-bound case (left panel) the EBU and NEB contributions turn out to be similar above the barrier but, as the incident energy decreases, the NEB drops faster than the EBU, making the latter dominant. This result corroborates the dominance of EBU observed in breakup experiments with halo nuclei [20,26]. Conversely, for the tightly bound case (right panel), the NEB dominates in the whole energy range.

These results confirm the strong sensitivity of the relative importance of EBU and NEB on the incident energy as well as on the separation energy. In particular, for halo nuclei, we expect a dominance of EBU at energies around and below the barrier, whereas for tightly bound nuclei we expect a dominance of NEB for all energies.

We have analysed also the post-prior equivalence of the IAV formula for the NEB cross sections. This is shown in the left panels of Fig. 5 for ($^{6}$Li,$\alpha X$) and in the right panels for ($^{6}$Li,$dX$). The thick solid and dashed lines are, respectively, the post-form and prior-form results of the IAV model. We see that the post-prior equivalence is accurately fulfilled. We have also included in this figure the three contributions of the prior-form IAV formula (10), namely, UT (thin solid line), NO (dotted line) and IN (dot-dashed line). It is interesting to observe that, in the ($^{6}$Li,$\alpha X$) case, the three contributions are of similar magnitude. By contrast, in the ($^{6}$Li,$dX$) case, the NO term is very small and hence the NEB cross section is largely dominated by the UT

**Table 1** Decomposition of the reaction cross section for the $^6$Li+$^{209}$Bi reaction at 30 and 38 MeV: elastic breakup (EBU), non-elastic breakup (NEB), complete fusion (CF), and their sum. All the cross sections are in mb

| $E_{lab}$ (MeV) | EBU (CDCC) | NEB ($^6$Li, $\alpha$X) (IAV) | NEB ($^6$Li, $d$X) (IAV) | CF Exp. [16] | EBU+NEB+CF | $\sigma_{reac}$ (OM) | $\sigma_{reac}$ (CDCC) |
|---|---|---|---|---|---|---|---|
| 30 | 38.3 | 185 | 15.3 | $37.7 \pm 1.3$ | 276 | 296 | 287 |
| 38 | 98.7 | 444 | 162 | $451 \pm 8$ | 1156 | 1151 | 1127 |

term. In this particular case, the IAV and UT models give compatible results. Recalling the definition of the NO term, Eq. (12), we may expect that this term will be more important when the projectile ($a$) and ejectile ($b$) are similar. Applied to the present case, we expect this term to be larger for ($^6$Li,$\alpha X$) than for ($^6$Li,$dX$), as testified by our numerical results.

It is enlightening to compare the different breakup contributions with the reaction cross section. This is shown in Table 1 for the two energies considered above (30 and 38 MeV). The listed values correspond to the total EBU (CDCC) and NEB (IAV) cross sections. For completeness, we include also in this Table the experimental CF cross section reported by Dasgupta et al. [16]. The sum of these contributions is compared at each energy with the reaction cross section, calculated from the optical model calculation with the Cook potential [12] as well as with the CDCC calculation (last two columns). It is seen that the sum EBU+NEB+CF is remarkably close to the reaction cross section. We believe that this is a very stringent test of consistency of the IAV model investigated here.

## 5 Open Problems and Possible Extensions

As mentioned in the previous section, all the inclusive breakup calculations performed so far rely on the DWBA approximation, i.e., they represent the exact scattering wave function by the product of a elastic scattering distorted wave ($\chi_a^{(+)}$) times the projectile and target ground-state wave functions, i.e., Eq. (4). The distorted wave $\chi_a^{(+)}$ is meant to include, in an effective way, all possible couplings affecting the elastic scattering of $a + A$. This includes, for instance, the excitation of the projectile and/or target. As occurs in other coupled-channels (CC) problems, it may happen that these intermediate states, which may also lead to NEB, need to be incorporated explicitly. Some examples are given below:

(i) If collective excitations to some states of the projectile or target are strong, one may include them explicitly using a CC approximation for $\Psi^{(+)}$. Notice that, in this case, Eq. (6) becomes formally analogous to that appearing in the standard CCBA method.
(ii) For very weakly-bound projectiles, the effect of breakup in the incident channel is important. In this case, $\Psi^{(+)}$ can be approximated by a CDCC wave function. This corresponds in fact to the three-body model proposed by Austern et al. [2]. Although the CDCC method is widely used nowadays, its implementation in the IAV formalism is not straightforward. Nevertheless, with the present computational capabilities, this should be feasible at least for specific cases.
(iii) The CDCC wave function contains in general many terms so the evaluation of the source term of the inhomogeneous Eq. (6) will be cumbersome. For incident energies of several tens of MeV per nucleon, one may invoke as an alternative the adiabatic approximation of Johnson and Soper [37]. This approximation will be valid when the average excitation energies of the projectile are small with respect the beam energy. Under this situation, the adiabatic wave function is known to reproduce well the full three-body wave function for small $b$-$x$ separations, which dominates the source term.
(iv) A more complete three-body description of the incident channel is given by the Faddeev wave function. This is the choice made in the formal works of Hussein and co-workers [31]. In practice, the solution of the Faddeev equations is too complicated for many practical applications and, even if this solution is available, its implementation in Eq. (6) will be very challenging.

We conclude this section by mentioning the possibility of applying the IAV theory to the calculation of incomplete fusion (ICF). As noted in the Introduction, ICF is part of the NEB cross section and, as such, is included in the double differential cross section (5). However, it is not straightforward how to isolate the ICF contribution from other sources of NEB associated with direct reactions (DR) of $x$ with the target, such as $x + A$ inelastic scattering. Assuming that one can split the imaginary part of $U_x$ as the sum of CN and

DR contributions, i.e., $W_x = W_x^{\rm DR} + W_x^{\rm CN}$, it is plausible to consider that the ICF cross section can be approximately calculated as

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\rm ICF} = -\frac{2}{\hbar v_a}\rho_b(E_b)\left\langle \psi_x^{\rm post}|W_x^{\rm CN}|\psi_x^{\rm post}\right\rangle. \tag{14}$$

The application and assessment of this formula for specific problems, such as surrogate reactions, is in progress.

## 6 Summary and Conclusions

In this contribution, we have addressed the problem of the calculation of inclusive breakup cross sections in reactions induced by weakly-bound projectiles, within the framework of the theory proposed by Ichimura, Austern and Vincent in the 1980s [2,33]. We have presented the post and prior formulas of this model, and discussed their formal and numerical equivalence. We have seen that the prior-form formula consists of three terms. One of these terms coincides with NEB formula proposed by Udagawa and Tamura (UT). The remaining terms, which arise from the non-orthogonality of the initial and final partitions, ensure the post-prior equivalence of the NEB cross sections.

We have applied the IAV and UT formulas to the $^{6}$Li+$^{209}$Bi reaction at 30 and 38 MeV. We have found that the experimental angular distributions of $\alpha$ particles are better reproduced by the sum EBU (calculated with CDCC) + NEB (IAV) than by the sum EBU+ NEB (UT), thus supporting the IAV theory. Furthermore, we have verified that the post and prior expressions of the NEB model provide essentially identical results, thus confirming the post-prior equivalence at a numerical level.

We have studied the dependence of the EBU and NEB contributions with the incident energy and the separation energy of the projectile. In most situations, we find a clear dominance of NEB. The EBU becomes only dominant for very small separation energies ($S_{\alpha d} \approx 0.5$ MeV), at near- and sub-Coulomb energies.

Finally, we have verified that the sum of the calculated EBU+NEB cross section and the experimental CF cross section is very close to the reaction cross section for the $^{6}$Li+$^{209}$Bi reaction, at the two considered energies. Since the reaction cross section imposes an strict upper limit for non-elastic processes, this result constitutes a robust consistency test of the theories considered here.

The results presented in this work, along with those presented in related works [11,40,41,46], indicate that the IAV model provides a reliable framework to calculate NEB cross sections in reactions induced by deuteron and $^{6}$Li projectiles. Possible applications to other systems and problems are currently under study.

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