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1978 J. Phys. G: Nucl. Phys. 4 1573

(http://iopscience.iop.org/0305-4616/4/10/008)

J. Phys. G: Nucl. Phys., Vol. 4, No. 10, 1978. Printed in Great Britain

A study of semiclassical approximations for heavy-ion transfer reactions

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Received 14 December 1977, in final form 3 May 1978

Abstract. Semiclassical concepts have been used to obtain a formula for determining angular distributions for heavy-ion transfer reactions starting from the expression of the transition matrix in DWBA. The distorted waves are expanded in partial waves and the radial wavefunctions approximated by the WKB approximation. Further approximations enable one of the partial-wave summations to be performed analytically. The transfer matrix element occurring in the resulting formula has been simplified to a one-dimensional integral by assuming a straight-line orbit for the relative motion. As an illustration, angular distributions have been calculated for the reactions ^{26}Mg(^{11}B, ^{10}B)^{27}Mg and ^{26}Mg(^{11}B, ^{10}Be)^{27}Al at 114 MeV laboratory energy and compared with experiment. The agreement is good.

1. Introduction

When two complex nuclei interact the wavelength associated with their relative motion is often short compared with characteristic nuclear dimensions. In such circumstances semiclassical concepts can be used in formulating reaction theories. Semiclassical theories are often more of an aid to the understanding of the physics rather than a means of performing systematic calculations, but it seems that they may be accurate enough to replace more exact but lengthy quantal theories in the analysis of experimental data.

One class of semiclassical theories for scattering problems is based on the WKB approximation in three dimensions. The wavefunction at a given point is given as a sum of semiclassical terms, each corresponding to a different classical trajectory ending up at the same point. This kind of theory is described in the work of Knoll and Schaeffer (1976). Derivations based on the Feynman path-integral method are similar in character (cf Koeling and Malfliet 1975). In theories based on classical trajectories it is not necessary to make any partial-wave expansion of the scattering wavefunction.

Another approach for calculating reaction cross sections is based on the distorted-wave Born approximation (DWBA). A formula for the total reaction amplitude can be written as a sum of contributions from different partial waves. In the work of Landowne *et al* (1976), the radial matrix elements in this formula are calculated

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0305-4616/78/0010-1573 $03.00

using WKB wavefunctions and then the contributions of different partial waves are summed numerically. Broglia *et al* (1974) make some further approximations to sum the partial-wave series and these authors are able to established a link between the partial-wave method and the classical trajectory method. This partial-wave method leads to a formula for the reaction amplitude $f(\alpha, \beta)$ of the form

$$f(\alpha, \beta) \alpha \sum_{l} (2l + 1)^{1/2} \eta(\tau, l) Y_{lm}(\alpha, \beta) \tag{1}$$

where $\eta(\tau, l)$ is a partial-wave amplitude containing a damping factor which takes into account absorption into other channels. In equation (1), m is the z component of transferred angular momentum and τ stands for various other quantum numbers characterising the reaction. In some works simple parametrisations of the partial-wave amplitudes $\eta(\tau, l)$ have been used (Strutinsky 1973, Frahn 1975).

In this paper we give a derivation of equation (1) for transfer reactions when one nucleon or an equivalent cluster of several nucleons is transferred between interacting nuclei. Our formulation starts with the expression for the transition matrix in the DWBA. The approach has some similarities with the work of Broglia *et al* (1974) and Landowne *et al* (1976), but differs in several respects. We make a partial-wave expansion and calculate the radial matrix elements using the method of Landowne *et al* (1976). The partial-wave series contains a double sum over the initial and final orbital angular momentum quantum numbers. We use an approximation to make one of these summations analytically. Our aim is the same as that of Broglia *et al* (1974), but the method we use is much simpler. This is one of the new features of the present work. The final expression for the amplitude has the form of equation (1) and the amplitude $\eta(\tau, l)$ is given by a time integral over a classical orbit appropriate to the final state of the system. The method involves an expansion about the final orbit and can be used only in situations where energy and angular momentum transfer are not too large. This part of the work is described in §§3, 4 and 5 of the paper.

Recoil effects are included in the calculation by an approximation in the initial and final distorted waves before the partial-wave expansion is made. The method is described in §§2 and 6.

The partial-wave transfer amplitude is evaluated in §7. It is given by a time integral of a matrix element of a potential between the initial and final single-particle states, and involves a four-dimensional integral (one over the time variable and three over space variables). The most important contribution to the time integral comes from the part of the classical orbit near the point of closest approach between the interacting nuclei. For high incident energies this part of the orbit can be approximated by a straight line tangential to the orbit. Then the time integral gives a δ function in the tangential component of transferred momentum and the four-dimensional integral reduces to a two-dimensional one in momentum space. One of these is an angular integration and can be made analytically so that the final expression for $\eta(\tau, l)$ involves only one-dimensional numerical integration. This reduction is another novel feature of the present work.

As an illustration, the method is applied to the study of the reactions ^{26}Mg(^{11}B, ^{10}B)^{27}Mg and ^{26}Mg(^{11}B, ^{10}Be)^{27}Al performed at 114 MeV laboratory energy. The angular distributions are compared with experiment (Paschopoulos *et al* 1975). Spectroscopic factors agree well with those obtained from a conventional DWBA analysis. In the range of angular momentum where $\eta(\tau, l)$ is large, the transfer

amplitude is seen to have an exponential dependence on l. We give a qualitative argument which leads to this result and suggest an estimate for the decay constant of the exponential.

2. Transition amplitude in DWBA

Consider the reaction

$$a_1 + c_2 \rightarrow c_1 + a_2$$

or

$$(c_1 + x) + c_2 \rightarrow c_1 + (c_2 + x)$$

where a particle x is transferred between two cores c_1 and c_2. The starting point is the DWBA expression for the transition amplitude (Dodd and Grieder 1969, Austern *et al* 1964):

$$T_{fi} = \int \chi_f^{(-)*}(\boldsymbol{k}_f, \boldsymbol{r}_f)\psi_2^*(\boldsymbol{r}_2)\Delta V(\boldsymbol{r}_2, \boldsymbol{s})\psi_1(\boldsymbol{r}_1)\chi_i^{(+)}(\boldsymbol{k}_i, \boldsymbol{r}_i)\,\mathrm{d}^3\boldsymbol{r}_2\,\mathrm{d}^3\boldsymbol{r}_f. \tag{2}$$

Here ψ_1, ψ_2 are the bound-state wavefunctions of the transferred particle x in the initial and final nuclei a_1 and a_2; $\chi_i^{(+)}$, $\chi_f^{(-)}$ are the distorted waves which describe the relative motion of the nuclei in the initial and final channels. ΔV is the interaction responsible for the transition. The coordinate system is illustrated in figure 1. The coordinates $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_f$, $\boldsymbol{r}_i$ and $\boldsymbol{s}$ are related by

$$\boldsymbol{r}_f = \boldsymbol{s} - \mu_2\boldsymbol{r}_2 \qquad \boldsymbol{r}_i = (1 - \mu_1)\boldsymbol{s} + \mu_1\boldsymbol{r}_2 \tag{3}$$

where

$$\mu_\alpha = m_x/(m_x + m_{c_\alpha}) \qquad \alpha = 1, 2$$

and m_x, m_{c_1} and m_{c_2} are the masses of the transferred particle and the cores c_1 and c_2 respectively.

The expression (2) for T_{fi} is a six-dimensional integral. It can be simplified by making an approximation based on a semiclassical approach similar to the one used

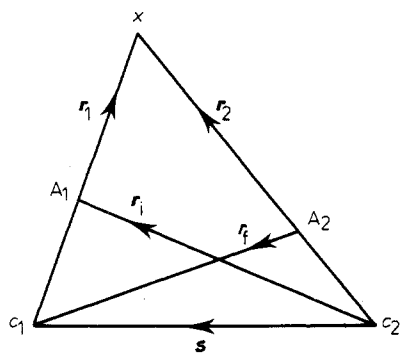

Figure 1. Relation between the various coordinates appearing in equation (2). A_1 denotes the centre of mass of c_1 and x while A_2 denotes that of c_2 and x.

by Dodd and Greider (1969). The WKB approximation for the distorted waves $\chi_f^{(-)}$ is (Glauber 1964, Schiff 1968)

$$\chi_f^{(-)}(\boldsymbol{r}_f) = A_f(\boldsymbol{r}_f) \exp[(i/\hbar)S_f(\boldsymbol{r}_f)] \tag{4}$$

where $S_f(\boldsymbol{r}_f)$ is a classical action function satisfying a Hamilton–Jacobi equation with boundary condition appropriate for a relative momentum $\boldsymbol{k}_f$ after the collision, and $|A_f^2|$ is the corresponding classical density function (Austern 1970). In the semiclassical limit, when $\hbar$ is small, the most rapidly varying term in equation (4) is $S_f/\hbar$. We expand this in a Taylor series about $\boldsymbol{s}$ up to first order in $(\boldsymbol{r}_f - \boldsymbol{s})$ to give

$$S_f(\boldsymbol{r}_f) \simeq S_f(\boldsymbol{s}) + (\boldsymbol{r}_f - \boldsymbol{s}) \cdot \boldsymbol{p}_f(\boldsymbol{s})$$

where

$$\boldsymbol{p}_f(\boldsymbol{s}) = \left(\frac{\partial S_f}{\partial \boldsymbol{r}_f}\right)_{\boldsymbol{r}_f = \boldsymbol{s}}$$

is the local momentum at $\boldsymbol{s}$ on the classical orbit through $\boldsymbol{s}$ which has relative momentum $\boldsymbol{k}_f$ after the collision. With this approximation

$$\chi_f^{(-)}(\boldsymbol{r}_f) \simeq \chi_f^{(-)}(\boldsymbol{s}) \exp[(i/\hbar)(\boldsymbol{r}_f - \boldsymbol{s}) \cdot \boldsymbol{p}_f(\boldsymbol{s})]. \tag{5}$$

Here we have assumed that A_f is slowly varying compared with $\exp[(i/\hbar)S_f]$ so that $A_f(\boldsymbol{r}_f) \simeq A_f(\boldsymbol{s})$. The analogous expression for $\chi_i^{(+)}$ is

$$\chi_i^{(+)}(\boldsymbol{r}_i) \simeq \chi_i^{(+)}(\boldsymbol{s}) \exp[(i/\hbar)(\boldsymbol{r}_i - \boldsymbol{s}) \cdot \boldsymbol{p}_i(\boldsymbol{s})] \tag{6}$$

where $\boldsymbol{p}_i(\boldsymbol{s})$ is evaluated on a classical orbit with asymptotic relative momentum $\boldsymbol{k}_i$ before the collision.

Substituting equations (5) and (6) into (1) and changing the variables of integration from $\boldsymbol{r}_2$, $\boldsymbol{r}_f$ to $\boldsymbol{s}$, $\boldsymbol{r}_2$, the six-dimensional integral can be written in terms of a form factor $G(\boldsymbol{s})$

$$T_{fi} = \int \chi_f^{(-)*}(\boldsymbol{s}) G(\boldsymbol{s}) \chi_i^{(+)}(\boldsymbol{s}) \, d^3\boldsymbol{s} \tag{7}$$

where

$$G(\boldsymbol{s}) = \exp[-(i/\hbar)\mu_1 \boldsymbol{p}_i \cdot \boldsymbol{s}] \int \psi_2^*(\boldsymbol{r}_2) \Delta V(\boldsymbol{r}_2, \boldsymbol{s}) \psi_1(\boldsymbol{r}_2 - \boldsymbol{s}) \exp[(i/\hbar)\boldsymbol{p} \cdot \boldsymbol{r}_2] \, d^3\boldsymbol{r}_2 \tag{8}$$

with

$$\begin{aligned} \boldsymbol{p}(\boldsymbol{s}) &= \mu_1 \boldsymbol{p}_i + \mu_2 \boldsymbol{p}_f \\ &= m_x \boldsymbol{v}(\boldsymbol{s}) \end{aligned}$$

where $\boldsymbol{v}(\boldsymbol{s})$ is an average velocity of relative motion at the point $\boldsymbol{s}$. Equations (7) and (8) are very close to expressions given by Dodd and Greider (1969). There are some differences because the approximations (5) and (6) to the distorted waves are not identical to the ones used by Dodd and Greider. Expression (7) for T_{fi} looks like a no-recoil approximation to DWBA. Effects of recoil are, in fact, included in the phase factors in equation (8) for $G(\boldsymbol{s})$ as discussed by Dodd and Greider (1969).

3. Partial-wave expansion

To evaluate equation (7) we make a partial-wave expansion of $\chi_i^{(+)}$ and $\chi_f^{(-)}$ as follows:

$$\chi^{(-)*}(\boldsymbol{k}, \boldsymbol{s}) = \chi^{(+)}(-\boldsymbol{k}, \boldsymbol{s}) = \frac{4\pi}{ks} \sum_{lm} \mathrm{i}^{-l} \mathrm{e}^{\mathrm{i}\delta(l)} f_l(s)\, Y_{lm}(\hat{\boldsymbol{k}})\, Y_{lm}^*(\hat{\boldsymbol{s}}). \tag{9}$$

In equation (9), $\delta(l)$ is the sum of the nuclear and Coulomb elastic-scattering phase-shifts and $f_l(s)$ is a radial wavefunction.

Choosing a coordinate system so that $\boldsymbol{k}_i$ defines the z axis and (α, β) are the polar angles of the direction $\boldsymbol{k}_f$, equation (7) yields

$$T_{fi} = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \sum_{l_i l_f m_f} (2l_i + 1)^{1/2} \exp\left[\mathrm{i}(\delta^i(l_i) + \delta^f(l_f))\right] \mathrm{i}^{l_i - l_f}\, Y_{l_f m_f}(\alpha, \beta)\, I(l_i, l_f, m_f) \tag{10}$$

where

$$I(l_i, l_f, m_f) = \frac{1}{\sqrt{k_i k_f}} \int_0^\infty f_{l_f}(s)\, G_{l_i l_f}^{m_f}(s) f_{l_i}(s)\, \mathrm{d}s \tag{11}$$

and

$$G_{l_i l_f}^{m_f}(s) = \int\int Y_{l_f m_f}^*(\theta, \phi)\, G(s, \theta, \phi)\, Y_{l_i 0}(\theta, \phi) \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi. \tag{12}$$

In equation (12), (s, θ, ϕ) are the polar coordinates of the vector $\boldsymbol{s}$.

In a heavy-ion reaction the dominant contributions to the sum (10) are expected to arise for large values of l_i and l_f. Hence we substitute the asymptotic forms of Y_{lm} in equation (12):

$$Y_{lm}(\theta, \phi) \simeq \frac{(-1)^m \mathrm{e}^{\mathrm{i}m\phi}}{\pi(\sin\theta)^{1/2}} \cos\left[(l + \tfrac{1}{2})\theta - \tfrac{1}{4}\pi + \tfrac{1}{2}m\pi\right]. \tag{13}$$

This gives

$$G_{l_i l_f}^{m_f}(s) \simeq \frac{(-1)^{m_f}}{2\pi^2} \int_0^{2\pi} \int_0^{\pi} \mathrm{e}^{-\mathrm{i}m_f\phi} G(s, \theta, \phi) \{\cos\left[(l_f - l_i)\theta + \tfrac{1}{2}m_i\pi\right] + \cos\left[(l_f + l_i + 1)\theta + (m_f - 1)\tfrac{1}{2}\pi\right]\} \mathrm{d}\theta\, \mathrm{d}\phi. \tag{14}$$

As the second term in the integrand of equation (14) is rapidly oscillating, it is expected to give a small contribution. Neglecting it gives

$$G_{l_i l_f}^{m_f}(s) \simeq \frac{(-1)^{m_f}}{2\pi^2} g(s, l_f - l_i, m_f) \tag{15}$$

where

$$g(s, L, M) = \int_0^{2\pi} \int_0^{\pi} \mathrm{e}^{-\mathrm{i}M\phi} G(s, \theta, \phi) \cos\left(L\theta + \tfrac{1}{2}M\pi\right) \mathrm{d}\theta\, \mathrm{d}\phi. \tag{16}$$

Equation (16) shows that $g(s, L, M)$ are the coefficients of a Fourier expansion of $G(s, \theta, \phi)$:

$$G(s, \theta, \phi) = \frac{1}{\pi^2} \sum_{L=0}^{\infty} \sum_{M=-\infty}^{+\infty} g(s, L, M) \cos\left(L\theta + \tfrac{1}{2}M\pi\right) \mathrm{e}^{\mathrm{i}M\phi}. \tag{17}$$

The functions $\cos(L\theta + \frac{1}{2}M\pi)\, e^{iM\phi}$ are a natural set for expanding $G(s, \theta, \phi)$ because, like $G(s, \theta, \phi)$, they are invariant with respect to the transformation $(\theta, \phi) \rightarrow (-\theta, \phi + \pi)$. Throughout this work we make the physical assumption that transferred angular momenta are small compared with the total angular momenta of relative motion of the interacting nuclei. This means that only small values of L and M contribute to the sum (17) or, equivalently, that $G(s, \theta, \phi)$ is a slowly varying function of θ and ϕ.

Equation (15) defining $g(s, L, M)$ gives the symmetry relation

$$g(s, -L, M) = (-1)^M g(s, L, M) \tag{18}$$

and, by using this relation, equation (17) can also be written as

$$G(s, \theta, \phi) = \frac{1}{2\pi^2} \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} g(s, L, M) \exp\{i[L\theta + M(\phi + \tfrac{1}{2}\pi)]\}. \tag{19}$$

4. WKB approximation for radial matrix elements

The next step is to make a semiclassical evaluation of the radial matrix element (11). We use WKB wavefunctions for $f_l(s)$ and follow the approach of Landowne *et al* (1976) to obtain

$$I(l_i, l_f, m_f) = \frac{(-1)^{m_f}}{8\pi^2} \int_C ds\, \frac{g(s, l_f - l_i, m_f)\, e^{i\Delta\phi(s)}}{\sqrt{k_f(s)}\sqrt{k_i(s)}} \tag{20}$$

where

$$\Delta\phi(s) = \int_{s_{0f}}^{s} k_f(s')\, ds' - \int_{s_{0i}}^{s} k_i(s')\, ds' \tag{21}$$

and

$$k_{i(f)}(s) = \frac{1}{\hbar}[2\mu(E - V_{i(f)})]^{1/2} \tag{22}$$

where $V_{i(f)}(s)$ is the sum of the Coulomb, nuclear and centrifugal potentials in the initial (final) state. The turning points $s_{0i(f)}$ are zeros of $E - V_{i(f)}$. The integration is over a contour C in the complex s plane which circumvents the two turning points s_{0i} and s_{0f} (figure 2). E is the centre-of-mass kinetic energy.

For a complex optical potential there are several turning points, each of which is complex. Malfliet *et al* (Malfliet 1975, Koeling and Malfliet 1975) and Knoll and Schaeffer (1976) have shown that the optical-model scattering phases are well approximated by the WKB phases calculated with respect to the outermost complex turning point when the absorption is strong. Consequently we shall use the results corresponding to this outermost turning point for our calculations.

Equation (20) for $I(l_i, l_f, m_f)$ in the present paper is somewhat different to the analogous formula in the paper of Landowne *et al* (1976), though the method of derivation is identical. The differences occur because in the present paper we approximate recoil corrections by equation (5) before making a partial-wave expansion. Landowne *et al* made the corresponding approximations in each partial wave separately.

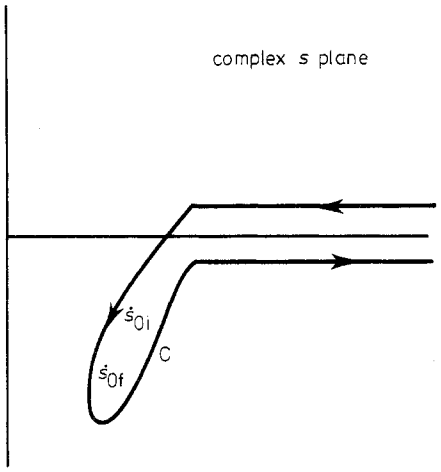

Figure 2. The contour C in the complex s plane used for the integration of equation (20).

5. Expansion about the final orbit

Each of the quantities $\not{k}_i(s)$, $\not{k}_f(s)$ in equation (20) correspond to a classical orbit with particular values of E, l, $V(s)$. If the orbits are not too different in the initial and final channels, we may expand about some average orbit. In this section we expand about the final orbit.

With this assumption of well matched orbits, the turning points s_{0f}, s_{0i} are close together and we may take them to be coincident. We also replace $\not{k}_i(s)$ by $\not{k}_f(s)$ in the denominator of equation (20), which may then be written as

$$I(l_i, l_f, m_f) = I^+(l_i, l_f, m_f) + I^-(l_i, l_f, m_f) \tag{23}$$

with

$$I^{\pm}(l_i l_f m_f) = \frac{(-1)^{m_f}}{8\pi^2} \int_{s_{0f}}^{\infty} \mathrm{d}s \, \frac{g(s, l_f - l_i, m_f)\, \mathrm{e}^{\pm \mathrm{i}\Delta\phi(s)}}{\not{k}_f(s)} \tag{24}$$

where the integrals are taken along the branch cut starting at the branch point s_{0f} (figure 2). It is convenient now to change the integration variable from s to time using the equation $s = s_f(t)$ of the final orbit:

$$\mathrm{d}s/\not{k}_f(s) = (\hbar/\mu_f)\, \mathrm{d}t.$$

Taking t real corresponds to a particular choice of the contour C. The integrand in equation (20) has a branch cut extending from s_{0f} to infinity, the two sides of the branch cut corresponding to $t > 0$ and $t < 0$ respectively while s_{0f} corresponds to $t = 0$. Thus I corresponds to a time integral between the limits $(-\infty, +\infty)$, while I^+ and I^- correspond to time integrals between the limits $(0, +\infty)$ and $(-\infty, 0)$ respectively. The first-order expression for $\Delta\phi(t)$ is

$$\Delta\phi(t) = \Delta\phi_0(t) - (l_f - l_i)\bar{\theta}_f(t) \tag{25}$$

with

$$\Delta\phi_0(t) = \frac{1}{\hbar}(E_f - E_i)t + \frac{1}{\hbar}\int_0^t (U_i(s_f(t)) - U_f(s_f(t)))\,dt$$

$$+ \frac{1}{2\hbar}(\mu_f - \mu_i)\int_0^t (\dot{s}_f(t))^2\,dt \tag{26}$$

where $U_{i(f)}(s)$ is the Coulomb plus nuclear potential in the initial (final) state. The function $\bar{\theta}_f(t)$ gives the azimuthal angle in the scattering plane measured from its symmetry axis for the final orbit. It is positive for $t > 0$ and negative for $t < 0$.

We also expand the phase $\delta^i(l_i)$ about l_f in a Taylor series and retain terms up to first order in $l_f - l_i$:

$$\delta^i(l_i) \simeq \delta^i(l_f) + \tfrac{1}{2}(l_i - l_f)\Theta_i(l_f)$$

where

$$\Theta_i(l_f) = [2d(\delta^i(l)/dl)]_{l=l_f}$$

corresponds to the classical deflection function for the initial orbit evaluated at $l = l_f$. In first order $\Theta_i(l_f)$ may be replaced by $\Theta_f(l_f)$. We further replace $(2l_i + 1)^{1/2}$ by $(2l_f + 1)^{1/2}$ as this factor does not occur in an exponent and is slowly varying.

Collecting these various approximations together, equation (10) may be written as

$$T_{fi} = \frac{(4\pi)^{3/2}}{\sqrt{k_f k_i}}\sum_{l_f}(2l_f + 1)^{1/2}\exp[i(\delta^i(l_f) + \delta^f(l_f))]K_{l_f}(\alpha, \beta) \tag{27}$$

where

$$K_{l_f}(\alpha, \beta) = (-1)^m \frac{\hbar}{\mu_f}\int_{-\infty}^{+\infty} dt\, e^{-i\Delta\phi_{0f}(t)}\, F_{l_f}(\theta_f(t), \alpha, \beta) \tag{28}$$

with

$$F_l(\theta, \alpha, \beta) = \frac{1}{8\pi^2}\sum_{l'm'}\exp[i(l - l')\theta]\, g(s, l - l', m')\, Y_{lm'}(\alpha, \beta) \tag{29}$$

and

$$\theta_f(t) = -\bar{\theta}_f(t) + \tfrac{1}{2}\Theta_f(l_f) + \tfrac{1}{2}\pi. \tag{30}$$

The quantity $\theta_f(t)$ is the angular position on the final orbit and $\bar{\theta}_f(t)$ is the angle defined in equation (25). The relation between the different angles is illustrated in figure 3.

The remaining step is to evaluate the function $F_l(\theta, \alpha, \beta)$ and the method is to use the Fourier expansion (19) to evaluate the summation in equation (29). To do this, it is necessary to specify the quantum numbers of the initial and final nuclear states more explicitly. Let (l_1, λ_1) and (l_2, λ_2) be the orbital angular momentum quantum numbers of the initial and final bound states ψ_1 and ψ_2. The function $G(s, \theta, \phi)$ defined in equation (8) depends explicitly on the polar angles (θ, ϕ) of the vector s. It also has an implicit dependence on the scattering angles α, β through the momentum functions $p_i(s)$ and $p_f(s)$ coming from the recoil corrections. In general we write

$$G(s, \theta, \phi) \equiv G(s, \theta, \phi, \beta)$$

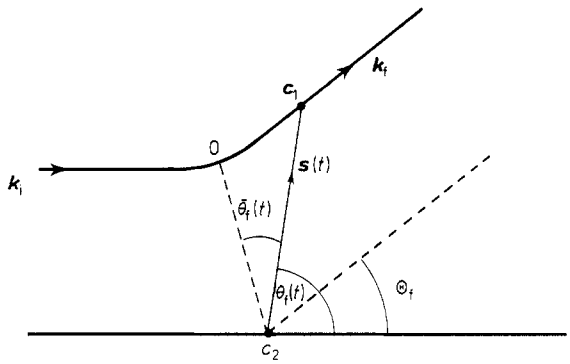

Figure 3. Relation between the various angles given in equation (30).

to show the β dependence explicitly. Then rotational invariance about the z axis gives

$$G(s, \theta, \phi, \beta) = \exp[i(\lambda_1 - \lambda_2)\phi]G(s, \theta, 0, \beta - \phi). \tag{31}$$

For forward-angle scattering the component of $\boldsymbol{p}_f$ perpendicular to the incident direction is not very large so that the dependence of $G(s)$ on α and β can be neglected. This further approximation to recoil effects should be unsatisfactory for large-angle scattering. For this case the alternative approximation discussed in the appendix can be used.

With the forward-angle approximation the dependence of $G(s, \theta, \phi)$ on ϕ is contained in the exponential factor in equation (31). Then $g(s, l - l', m_f) = 0$ unless $m_f = \lambda_1 - \lambda_2$ and the summation (29) can be evaluated using equation (19) to give

$$F_l(\theta, \alpha, \beta) = \tfrac{1}{4}G(s, \theta, 0)\, i^{\lambda_2 - \lambda_1}\, Y_{l,\lambda_1 - \lambda_2}(\alpha, \beta). \tag{32}$$

The summation variable has been changed using $l' - l = L$ and the summation extended from $-\infty$ to $+\infty$. This is possible because we have assumed in §2 that $g(s, l - l', m')$ is small except for $l - l'$ near zero. Collecting equations (27), (28) and (32), we obtain

$$T_{fi} = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \frac{\hbar^2}{4\mu_f} \exp[i(\lambda_1 - \lambda_2)\tfrac{1}{2}\pi] \sum_l (2l + 1) \exp[i(\delta^i(l) + \delta^f(l))]$$
$$\times A_l(\lambda_2, \lambda_1)\, Y_{l,\lambda_1 - \lambda_2}(\alpha, \beta) \tag{33}$$

with

$$A_l(\lambda_2, \lambda_1) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dt\, e^{-i\Delta\phi_0(t)} G(s(t), \theta(t), 0) \tag{34}$$

where the integral in equation (34) is to be taken along a classical orbit with $\phi = 0$ and orbital angular momentum l.

6. The semiclassical transfer amplitude

In this section we show that the $A_l(\lambda_2, \lambda_1)$ in equation (33) can be identified as a semiclassical transfer amplitude (Brink 1972, Broglia *et al* 1974, Anayas-Weiss *et*

al 1974, Pixton 1972). The phase $\Delta\phi_0$ is defined by equation (26). We shall assume that the most important contribution to $(U_f - U_i)$ is due to the Coulomb term and comes from parts of the orbit $s_f(t)$ near the point of closest approach, i.e.

$$U_f - U_i \approx \frac{(Z_{c_1} - Z_{c_2})Z_x e^2}{d}$$

where Z_{c_1}, Z_{c_2} and Z_x are the charges on the cores c_1, c_2 and the transferred particle x respectively, and d is the distance of closest approach. With this approximation the contribution of the first two terms in equation (26) is $Q_{\text{eff}}t/\hbar$ where

$$Q_{\text{eff}} = E_f - E_i + \frac{(Z_1 - Z_2)Z_x e^2}{d} \tag{35}$$

is the effective Q value (cf Brink 1972, Broglia *et al* 1974).

As $\Delta\phi_0$ is important only near the point of closest approach, we shall further assume that

$$(\dot{s}(t))^2 \simeq (\dot{s})^2_{t=0}$$

in the second term of equation (24). With these approximations

$$\Delta\phi_0 \approx -\frac{t}{\hbar}[\tfrac{1}{2}(\mu_f - \mu_i)\dot{s}^2 + Q_{\text{eff}}]. \tag{36}$$

As a next step we study the interaction potential ΔV (equations (2), (8)) in more detail. According to De Vries *et al* (1974) this consists of a sum of a nuclear part ΔV_n and a Coulomb part ΔV_C given in the prior representations as

$$\Delta V_n = V^n_{c_2 x}(\boldsymbol{r}_2) + V^n_{c_1 c_2}(\boldsymbol{s}) - U^{\text{opt},n}_{c_2 a_1}(\boldsymbol{r}_i) \tag{37}$$

$$\Delta V_C = V^C_{c_2 x}(\boldsymbol{r}_2) + V^C_{c_1 c_2}(\boldsymbol{s}) - U^{\text{opt},C}_{c_2 a_1}(\boldsymbol{r}_i) \tag{38}$$

where $V^n_{c_2 x} + V^C_{c_2 x}$ is the bound-state potential V_2, V_{c_1,c_2} is the core–core potential and U^{opt} is the optical potential in the final channel. There are analogous expressions for ΔV in the post representation. We follow De Vries *et al* and omit the last two terms in equation (37) but keep all the terms of equation (38). We approximate ΔV_C by

$$\Delta V_C \simeq V^C_{c_2 x}(\boldsymbol{r}_2) - \Delta\epsilon_2 \tag{39}$$

where

$$\Delta\epsilon_2 = (Z_{c_2} Z_x e^2)/d. \tag{40}$$

The replacement of $U^{\text{opt},C}_{c_2 a_1}(\boldsymbol{r}_i) - V^C_{c_1 c_2}(\boldsymbol{s})$ by $\Delta\epsilon_2$ can be justified by the following arguments. As the distance of closest approach in a peripheral collision is greater than the sum of the radii of the nuclei, the Coulomb interactions between them may be replaced by point-charge potentials

$$V^C_{c_1 c_2}(\boldsymbol{s}) - U^{\text{opt},C}_{c_2 a_1}(\boldsymbol{r}_i) \simeq \frac{Z_{c_1} Z_{c_2} e^2}{|\boldsymbol{s}|} - \frac{Z_{c_2} Z_{a_1} e^2}{|\boldsymbol{r}_i|} \tag{41}$$

If the mass of the transferred particle is much less than that of the cores then $|\boldsymbol{r}_f|$ in the above equation may be replaced by $|\boldsymbol{s}|$. Also we expect the most important contributions to the matrix element (8) to come from $|\boldsymbol{s}| \simeq d$. This gives equation (40). Collecting these results gives $\Delta V(\boldsymbol{r}_2) = V_2(\boldsymbol{r}_2) - \Delta\epsilon_2$ in the prior representation.

Substituting equations (8) and (36) into equation (34) and assuming that

$$\boldsymbol{p}_i \approx \mu_i \dot{\boldsymbol{s}}/\hbar$$

in equation (8), we obtain

$$A_l(\lambda_2, \lambda_1) = \frac{1}{\hbar} \int_{-\infty}^{\infty} U_{21}(t) \exp\left[(-\mathrm{i}t/\hbar)(Q_{\mathrm{eff}} + \tfrac{1}{2} m_x \dot{s}^2)\right] \mathrm{d}t \tag{42}$$

where

$$U_{21}(t) = \int \psi_2^*(\boldsymbol{r}_2)\, \Delta V(\boldsymbol{r}_2)\, \psi_1(\boldsymbol{r}_2 - \boldsymbol{s}(t)) \exp\left[(\mathrm{i}/\hbar) m_x \dot{\boldsymbol{s}} . \boldsymbol{r}_2\right] \mathrm{d}^3 r_2 . \tag{43}$$

The quantity $A_l(\lambda_2, \lambda_1)$ can be recognised as the semiclassical transfer amplitude of Brink (1972).

In the derivation of equation (33), quantities were expanded about the final orbit $\boldsymbol{s}_f(t)$. The expansions could also have been made about the initial orbit $\boldsymbol{s}_i(t)$ or about some average orbit $\boldsymbol{s}(t)$.

7. Evaluation of transfer amplitudes

To evaluate $A_l(\lambda_2, \lambda_1)$ in equation (42), we shall approximate the average path $\boldsymbol{s}(t)$ to be a straight-line trajectory:

$$\boldsymbol{s}_0(t) = \boldsymbol{d} + \boldsymbol{v}t \tag{44}$$

which is tangential to the orbit $\boldsymbol{s}(t)$ at its point of closest approach. Here d is the distance of closest approach, and $\boldsymbol{v}$ is the relative velocity at this point.

In equation (42) the orbit $\boldsymbol{s}(t)$ lies in the (x, z) plane. It is convenient to make a rotation about y so that the new z axis is parallel to $\boldsymbol{v}$. The transfer amplitude $A_l'(\lambda_2', \lambda_1')$ in this frame of reference is then related to $A_l(\lambda_2, \lambda_1)$ through the relation

$$A_l(\lambda_2, \lambda_1) = \sum_{\lambda_1' \lambda_2'} \mathscr{D}^{I_1}_{\lambda_1' \lambda_1}(0, \tfrac{1}{2}\alpha, 0)\, \mathscr{D}^{I_2}_{\lambda_2' \lambda_2}(0, -\tfrac{1}{2}\alpha, 0)\, A_l'(\lambda_2', \lambda_1') \tag{45}$$

where $\mathscr{D}^{I_i}_{\lambda_i' \lambda_i}$ are rotation matrices defined according to Brink and Satchler (1968).

$A_l'(\lambda_1', \lambda_2')$ can be evaluated by writing equation (43) in momentum space. Thus in the prior representation we have

$$A_l'(\lambda_2', \lambda_1') = \frac{1}{\hbar} \int\int \tilde{\psi}_2^*(\boldsymbol{k}_2) \left(\epsilon_1 - \Delta\epsilon_1 - \frac{\hbar^2}{2m_x} k_1^2\right) \tilde{\psi}_1(\boldsymbol{k}_1) \exp\left(-\mathrm{i}\boldsymbol{k}_1 . \boldsymbol{d}\right)$$
$$\times \exp\left[(-\mathrm{i}t/\hbar)(Q_{\mathrm{eff}} + \tfrac{1}{2} m v^2 + \hbar k_{1z} v)\right] \mathrm{d}^3 \boldsymbol{k}_1 \, \mathrm{d}t. \tag{46}$$

In equation (46) $\tilde{\psi}_1(\boldsymbol{k}_1)$ and $\tilde{\psi}_2(\boldsymbol{k}_2)$ are the momentum-space transforms of the bound-state wavefunctions $\psi_1(\boldsymbol{r})$ and $\psi_2(\boldsymbol{r})$. The quantity ϵ_1 is the energy of the state ψ_1, that is, the eigenvalue of

$$(T + V_1)\psi_1 = \epsilon_1 \psi_1,$$

and $\Delta\epsilon_1$ is the Coulomb correction given in equation (40). The wavevectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ in equation (46) are related by

$$\boldsymbol{k}_2 = \boldsymbol{k}_1 + m_x \boldsymbol{v}/\hbar. \tag{47a}$$

The time integration in equation (46) gives a δ function in k_{1z} so that

$$k_{1z} = -\frac{1}{\hbar v}(Q_{\text{eff}} + \tfrac{1}{2}mv^2) \qquad k_{2z} = -\frac{1}{\hbar v}(Q_{\text{eff}} - \tfrac{1}{2}mv^2). \tag{47b}$$

Because of the δ function, the integral over k_{1z} can be calculated easily. Hence the four-dimensional integral (46) can be reduced to a two-dimensional integral over k_{1x} and k_{1y}. This can be further reduced to a one-dimensional integral by first breaking up the $\psi_i(\boldsymbol{k})$ into radial parts $u_l(k)$ and angular parts consisting of spherical harmonics, then changing variables to cylindrical coordinates $k_\perp$, ϕ and finally integrating over ϕ. The expression obtained is given by

$$A_l'(\lambda_1', \lambda_2') = \frac{(2\pi)^2}{\hbar v} \mathrm{i}^{\lambda_2' - \lambda_1'} \int \tilde{u}_2^*(k_2)\, u_1(k_1)\, Y_{l_2\lambda_2'}^*(\theta_2, 0)\, Y_{l_1\lambda_1'}(\theta_1', 0)$$

$$\times \left(\epsilon_1 - \Delta\epsilon_1 - \frac{\hbar^2}{2m_x} k_1^2\right) J_{\lambda_1' - \lambda_2'}(k_\perp d)\, \mathrm{d}k_\perp . \tag{48}$$

Here $J_\mu(kd)$ with $\mu = \lambda_1' - \lambda_2'$ is a Bessel function. The quantities k_1, k_2, θ_1, θ_2, k_{1z} and ϕ are related as shown in figure 4. Equation (48) is a convenient expression for calculating the transfer amplitude because it involves only a one-dimensional integral over the variable $k_\perp$. The l dependence is contained in the distance of closest approach d. If the final orbit is approximated by a Rutherford orbit then d is given by

$$k_{\text{f}} d = n_{\text{f}} + [n_{\text{f}}^2 + (l + \tfrac{1}{2})^2]^{1/2}$$

where k_{f} is the final asymptotic wavenumber and n_{f} is the corresponding Sommerfeld parameter. The momentum-space radial wavefunctions $\tilde{u}_1$ and $\tilde{u}_2$ are required, but these are the same for each l and need only be calculated once. Equation (48) leads to the same expression for A_l' in both the post and prior representations because the relations (47) make

$$(\epsilon_1 - \Delta\epsilon_1 - \hbar^2 k_1^2/2m_x) = (\epsilon_2 - \Delta\epsilon_2 - \hbar^2 k_2^2/2m_x).$$

The effective Q value (35) is related to the binding energies and Coulomb energies by

$$Q_{\text{eff}} = \epsilon_2 - \epsilon_1 - (\Delta\epsilon_2 - \Delta\epsilon_1).$$

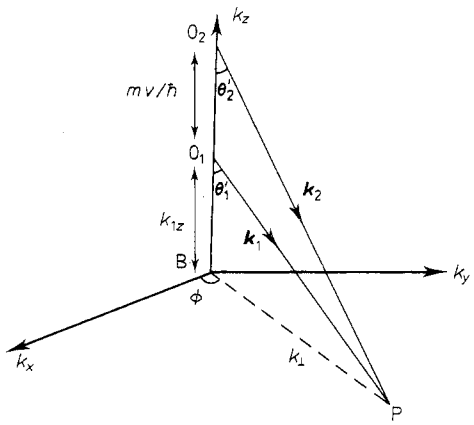

Figure 4. Relation between the coordinates appearing in equation (47).

The transfer amplitude $A_l(\lambda_2, \lambda_1)$ obtained by using equations (45)–(48) is substituted into equation (33) to give an expression for T_{fi}.

The differential cross section for the transfer of a particle from a definite state j_1 to a definite state j_2 is given by

$$\frac{d\sigma}{d\Omega} = \frac{2J_{a_2} + 1}{(2J_{c_2} + 1)(2j_1 + 1)(2j_2 + 1)} \frac{k_f}{k_i} \frac{\mu_i \mu_f}{(2\pi\hbar^2)^2} \sum_{n_1 n_2} |T(j_1 n_1, j_2 n_2)|^2 \qquad (49)$$

where

$$T(j_1 n_1, j_2 n_2) = \sum_{\lambda_1 \lambda_2 m_s} \langle l_1 \lambda_1, s m_s | j_1 n_1 \rangle \langle l_2 \lambda_2, s m_s | j_2 n_2 \rangle T_{fi}(\lambda_2, \lambda_1). \qquad (50)$$

J_{a_2}, J_{c_2} are the spins of the final nucleus and the core c_2 respectively; $\langle l_i \lambda_i, s m_s | j_i n_i \rangle$ are Clebsch–Gordan coefficients. The (λ_2, λ_1) dependence of T_{fi} has been included specifically in the notation.

8. Application to the reactions $^{11}B + {}^{26}Mg$

We have applied the formulae derived in §§2–7 to calculate angular distributions for the reactions $^{26}Mg(^{11}B, {}^{10}B)^{27}Mg$ and $^{26}Mg(^{11}B, {}^{10}Be)^{27}Al$ at 114 MeV laboratory energy. The transfer amplitudes $A_l(\lambda_2, \lambda_1)$ were calculated from equation (48). The radial-state wavefunctions $u_i(r)$ were obtained using a Woods–Saxon potential with a radius parameter $R = r_0 A^{1/3}$ ($r_0 = 1{\cdot}20$ fm, A = mass number of nucleus) and a diffusivity $a = 0{\cdot}65$. The depth of the potential was adjusted in each case to give the correct binding energy. The optical potential used to determine the elastic-scattering phase-shifts appearing in equation (33) was taken to have a Woods–Saxon form for both the real and imaginary parts, the parameters being given by

$$V_R = 35 \text{ MeV} \qquad V_I = 25 \text{ MeV}$$

$$a_R = 0{\cdot}8 \text{ fm} \qquad a_I = 0{\cdot}62 \text{ fm}$$

$$r_{0R} = 1{\cdot}066 \text{ fm} \qquad r_{0I} = 1{\cdot}216 \text{ fm}$$

$$R_{R(I)} = r_{0R(I)}(A_p^{1/3} + A_t^{1/3})$$

where $V_{R(I)}$, $a_{R(I)}$, $R_{R(I)}$ are the potential depth, diffusivity and radius parameter respectively for the real (imaginary) part of the potential. A_p, A_t are the mass numbers of the projectile and target respectively. The potentials in the initial and final channels differed only in the radius parameter $R_{R(I)}$.

8.1. Application to the reaction $^{26}Mg(^{11}B, {}^{10}B)^{27}Mg$

Angular distributions for the transfer of a neutron from the $1p_{3/2}$ state in ^{11}B to the ground and excited states in ^{27}Mg have been calculated. A detailed study of the l dependence of the transfer amplitude $A_l(\lambda_2, \lambda_1)$ and the product $A_l(\lambda_2, \lambda_1) \exp(-2\delta(l))$ (where $2\delta(l) = \delta^i(l) + \delta^f(l)$) appearing in equation (33) shows some interesting features which can be exploited to our advantage. Figure 5 shows that $A_l(\lambda_2, \lambda_1)$ decays to zero approximately exponentially as l increases for two

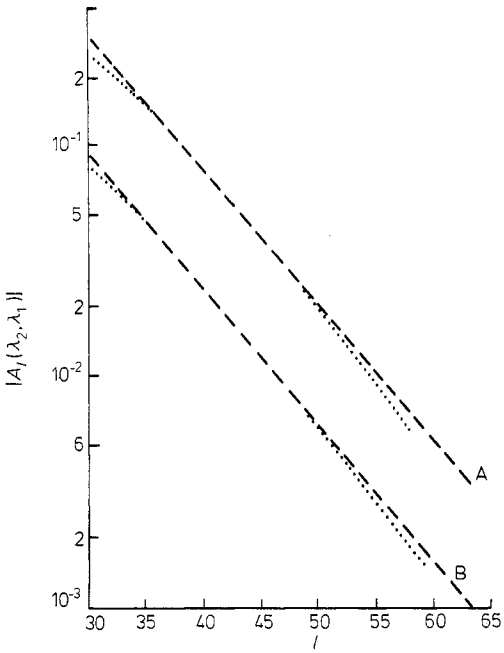

Figure 5. A comparison of $A_l(\lambda_2, \lambda_1)$ calculated from equations (48) and (51) for transfer of a neutron from a $1p_{3/2}$ state in ^{11}B to a $2s_{1/2}$ state in ^{27}Mg for (λ_2, λ_1) equal to (A) $(0, -1)$ and (B) $(0, 0)$. The dotted curve in each case corresponds to the calculation using equation (48) while the broken curve corresponds to the parametrised formula equation (51). The values of the parameters are $\Lambda_0 = 20{\cdot}9$, $\Delta = 7{\cdot}44$, $|K(0), -1)| = 1$, $|K(0, 0)| = 0{\cdot}268$.

particular final neutron states. This dependence was found for all final states studied and can be approximated quite well by the formula

$$A_l(\lambda_2, \lambda_1) \simeq K(\lambda_2, \lambda_1) \exp[(\Lambda_0 - l)/\Delta] \tag{51}$$

where $K(\lambda_2, \lambda_1)$, Λ_0 and Δ are parameters to be fitted. The parameter Δ turns out to be almost independent of λ_1, λ_2 so that the entire dependence of $A_l(\lambda_2, \lambda_1)$ on these quantum numbers is contained in $K(\lambda_2, \lambda_1)$. These parameters can be evaluated by calculating $A_l(\lambda_2, \lambda_1)$ from equation (48) for two suitable values of l. Figure 6 shows the product $A_l(\lambda_2, \lambda_1) \exp(-2\,\mathrm{Im}\delta(l))$ calculated exactly and also using the exponential approximation (51) for $A_l(\lambda_2, \lambda_1)$. The exponential dependence of A_l provides an upper cut-off to the summation in equation (33). A lower cut-off is provided by the imaginary part of the phase-shift $\delta(l)$ which causes the contribution from lower values of l to be damped out.

The l dependence of the amplitude A_l is contained in the distance of closest approach d. For a peripheral collision, equations (42) and (43) suggest that this dependence might be proportional to $\exp(-\gamma d)$ where $1/\gamma$ is a typical fall-off distance for a radial wavefunction. When the Sommerfeld parameter is small the relation between d and l for a Rutherford orbit leads to a dependence of A_l on l with the form of equation (51) with $\Delta = k_f/\gamma$. It remains to estimate a value for γ. For a neutron the asymptotic form of the radial wavefunction $u_1(r)$ is exponential with $\gamma_1 = (-\epsilon_1 2m_x/\hbar^2)^{1/2}$. The decay constant γ_2 for $u_2(r)$ is related to ϵ_2 in the same

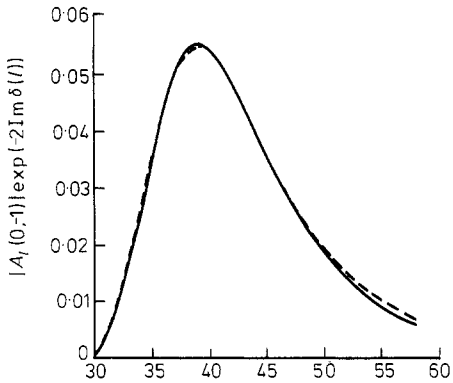

Figure 6. A comparison of the product $A_l(\lambda_2, \lambda_1)\exp(-2\,\mathrm{Im}\delta(l))$ calculated using equations (48) and (51) for the transfer of a neutron from a $1p_{3/2}$ state in ^{11}B to the $2s_{1/2}$ state in ^{27}Mg for (λ_2, λ_1) equal to $(0, -1)$. The full curve corresponds to the calculation using equation (48) while the broken curve corresponds to the parametrised formula equation (51).

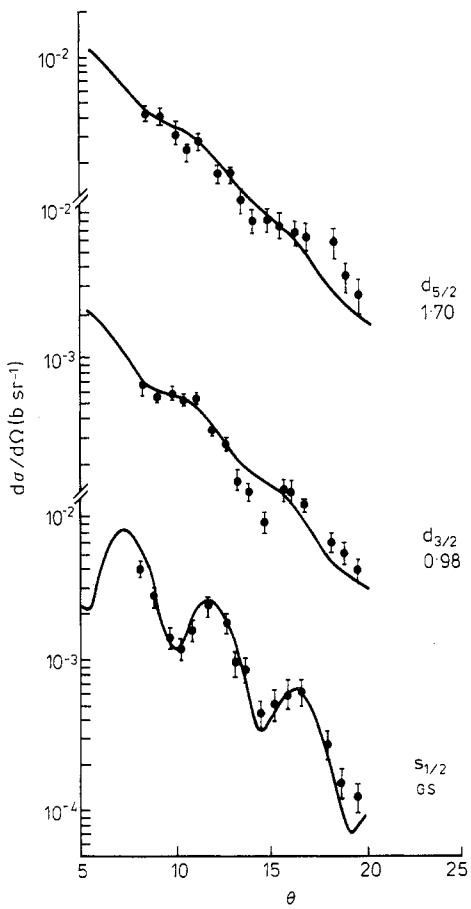

Figure 7. A comparison of the angular distributions obtained using equation (33) (full curves) with experiment (full circles) for the transfer of a neutron from the $1p_{3/2}$ state in ^{11}B to the ground and excited states in ^{27}Mg for the reaction $^{26}Mg(^{11}B, ^{10}B)^{27}Mg$ performed at 114 MeV laboratory energy.

Table 1.

Reaction	Excitation energy (MeV)	J^π	Spectroscopic factor LOLA	Spectroscopic factor Present calculation
$^{26}Mg(^{11}B, ^{10}B)^{27}Mg$	0·0	$\frac{1}{2}^+$	0·62	0·62
	0·98	$\frac{3}{2}^+$	0·50	0·37
	1·70	$\frac{5}{2}^+$	0·20	0·22
$^{26}Mg(^{11}B, ^{10}Be)^{27}Al$	0·0	$\frac{5}{2}^+$	0·15	0·19
	0·84	$\frac{1}{2}^+$	0·20	0·18
	6·48	$\frac{7}{2}^-$	0·20	0·22

way. The constants γ_1 or γ_2 would be possible choices for γ. Because of the equality of the amplitudes in the post and prior representations, γ should depend symmetrically on quantities characterising the initial and final states. The discussion in § 7 suggests

$$\bar{\gamma}^2 = 2m_x(\Delta\epsilon_1 - \epsilon_1)/\hbar^2 + k_{1z}^2 = 2m_x(\Delta\epsilon_2 - \epsilon_2)/\hbar^2 + k_{2z}^2$$

as another possible choice for γ. The parameters of the reaction discussed in this section give $k_f/\bar{\gamma} = 6{\cdot}7$. This is in reasonable agreement with the value of $\Delta = 7{\cdot}4$ obtained from a numerical integration of equation (48) (cf figure 5).

The approximation of $A_l(\lambda_2, \lambda_1)$ by the formula (51) simplifies the calculation considerably. We used this procedure to evaluate $A_l(\lambda_2, \lambda_1)$ and calculated angular distributions. The results are illustrated in figure 7 together with the experimental data of Paschopoulos *et al* (1975). In each case the cross sections are normalised by the corresponding spectroscopic factors obtained by comparison with experiment and plotted on the same scale as the experimental results. The fit is seen to be generally good.

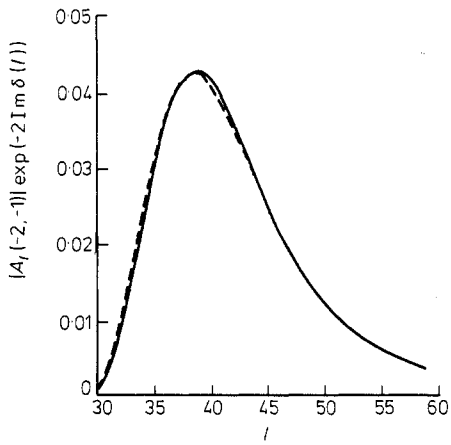

Figure 8. A comparison of the product $A_l(\lambda_2, \lambda_1)\exp(-2\,\mathrm{Im}\delta(l))$ calculated using equations (48) and (51) for the transfer of a proton from a $1p_{3/2}$ state in ^{11}B to a $1d_{5/2}$ state in ^{27}Al for (λ_2, λ_1) equal to $(-2, -1)$. The full curve corresponds to the calculations using equation (48) while the broken curve corresponds to the parametrised formula equation (51). The values of the parameters are $\Lambda_0 = 21{\cdot}18$, $\Delta = 6{\cdot}60$, $|K(-2, -1)| = 1$.

A comparison of the spectroscopic factors obtained from the present calculations is made with those obtained from the exact finite-range code LOLA of De Vries (1973) in table 1. The agreement is good.

8.2. Application to the reaction $^{11}B(^{26}Mg, ^{25}Al)^{10}Be$

The method outlined above was also used to obtain angular distributions for the transfer of a proton from a $p_{3/2}$ state to the ground and excited states of ^{27}Al at 114 MeV laboratory energy. As in the case of neutron transfer, the l dependence of $A_l(\lambda_2, \lambda_1)$ is represented quite accurately by equation (51). Figure 8 shows the product $A_l(\lambda_2, \lambda_1) \exp(-2\,\mathrm{Im}\delta(l))$ calculated exactly and with the approximation (51) for one particular final state. Angular distributions are compared with experiment (Paschopoulos *et al* 1975) in figure 9. The agreement is good.

9. Conclusions

The semiclassical formula obtained here provides a simple way for evaluating angular distributions. The approximation of the transfer amplitudes by a parametrised formula seems a promising one, as it greatly reduces the complexity of the calculation.

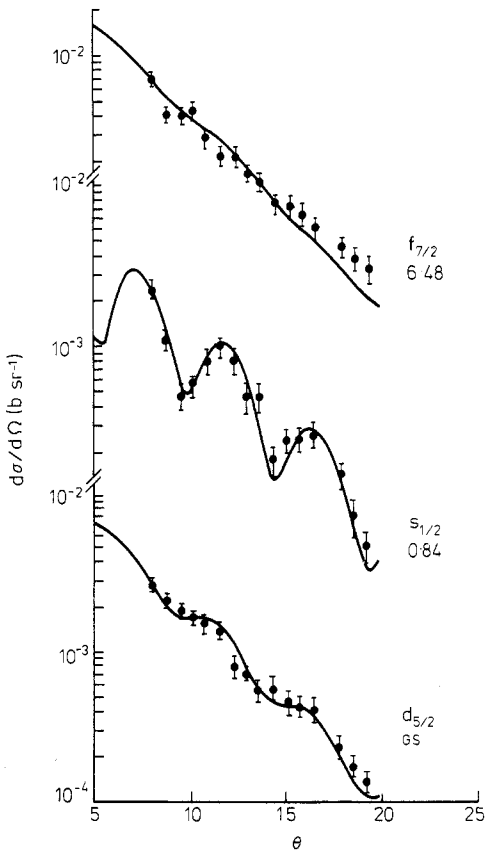

Figure 9. A comparison of the angular distributions obtained using equation (33) (full curves) with experiment (full circles) for the transfer of a proton from the $1p_{3/2}$ state in ^{11}B to the ground and excited states in ^{27}Al for the reaction $^{26}Mg(^{11}B, ^{10}Be)^{27}Al$ performed at 114 MeV laboratory energy.

This could be of advantage when studying reactions between heavy nuclei involving large angular momentum transfers.

An added merit of this formulation is the way in which the various components of the transition matrix factorise out and we can 'see' how the reaction is taking place physically.

The range of applicability of this formula is, however, restricted to well matched reactions, as this is a fundamental assumption in the derivation. Further, because of the number of approximations made, it is not very reliable for determining good spectroscopic factors, though it does give a reasonable estimate.

Acknowledgments

One of us (HH) would like to acknowledge financial support from the Association of Commonwealth Universities for the duration of her stay in Oxford.

Appendix

In deriving equation (33) we have assumed that the scattering takes place at forward angles, which led us to the expression (32) for the quantity $F_l(\theta, \alpha, \beta)$ defined by equation (29). For large-angle scattering we obtain a modified form of equation (33) by proceeding as follows. We replace $Y_{lm'}(\alpha, \beta)$ appearing in equation (29) by its asymptotic form (13) for large l and expand the cosine, leading to

$$F_l(\theta, \alpha, \beta) = \frac{1}{8\pi^2} \frac{1}{2\pi\sqrt{\sin\alpha}} \sum_{l'm'} (g(s, l - l', m') \exp[\mathrm{i}(l - l')\theta] \exp[\mathrm{i}m'(\beta + \tfrac{3}{2}\pi)]$$
$$\times \exp\{-\mathrm{i}[\tfrac{1}{4}\pi + (l + \tfrac{1}{2})\alpha]\} + g(s, l - l', m') \exp[\mathrm{i}(l - l')\theta]$$
$$\times \exp[\mathrm{i}m'(\beta + \tfrac{1}{2}\pi)] \exp\{\mathrm{i}[\tfrac{1}{4}\pi + (l + \tfrac{1}{2})\alpha]\}). \tag{A.1}$$

As before, we change the summation variable using $l' - l = L$ and extend the summation from $-\infty$ to $+\infty$. Then from equation (19) we get

$$F_l(\theta, \alpha, \beta) = \frac{1}{4} \frac{1}{2\pi\sqrt{\sin\alpha}} (\exp\{\mathrm{i}[(l + \tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\} G(s, \theta, \beta + \pi, \beta)$$
$$+ \exp\{-\mathrm{i}[(l + \tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\} G(s, \theta, \beta, \beta)). \tag{A.2}$$

Using equation (31), this becomes

$$F_l(\theta, \alpha, \beta) = \frac{1}{4} \frac{1}{2\pi\sqrt{\sin\alpha}} \exp[\mathrm{i}(\lambda_1 - \lambda_2)\beta] (\exp\{\mathrm{i}[(l + \tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\} G(s, \phi, \pi, 0)$$
$$+ \exp\{-\mathrm{i}[(l + \tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\} G(s, \theta, 0, 0)). \tag{A.3}$$

We may interpret this equation as giving the contributions from the two opposite sides of the scattering nucleus corresponding to $\phi = \pi$ and $\phi = 0$ respectively.

Substituting equation (A.3) into equation (28), which is put back into equation (27), we get the modified form of T_{fi} for large-angle scattering:

$$T_{fi} = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \frac{\hbar^2}{4\mu_f} \frac{\exp[i(\lambda_1 - \lambda_2)\beta]}{2\pi\sqrt{\sin\alpha}} \sum_l (2l+1)^{1/2} \exp[i(\delta^i(l) + \delta^f(l))]$$
$$\times (A_l^+(\lambda_2, \lambda_1) \exp\{i[(l+\tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\} + A_l^-(\lambda_2, \lambda_1)$$
$$\times \exp\{-i[(l+\tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\}) \tag{A.4}$$

where

$$A_l^+(\lambda_2, \lambda_1) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dt \exp(-i\Delta\phi_0(t))\, G(s(t), \theta(t), 0, 0) \tag{A.5}$$

and

$$A_l^-(\lambda_2, \lambda_1) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dt \exp(-i\Delta\phi_0(t))\, G(s(t), \theta(t), \pi, 0). \tag{A.6}$$

For forward-angle scattering we have $A_l^-(\lambda_2, \lambda_1) = (-1)^{\lambda_1 - \lambda_2} A_l^+(\lambda_2, \lambda_1)$ and equation (A.4) reduces to equation (33).

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